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Recursion relations in CFT and N=2 SYM theory

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Abstract: Based on prototypical example of Al.Zamolodchikov's recursion relations for the four point conformal block and using recently proposed Alday-Gaiotto-Tachikawa (AGT) conjecture, recursion relations are derived for the generalized prepotential of $\mathcal{N} = 2$ SYM with $f = 0, 1, 2, 3, 4$ (anti) fundamental or an adjoint hypermultiplets. In all cases the large expectation value limit is derived explicitly. A precise relationship between generic 1-point conformal block on torus and specific 4-point conformal block on sphere is established. In view of AGT conjecture this translates into a relation between partition functions with an adjoint and 4 fundamental hypermultiplets.

Keywords: Conformal Field Theory, Gauge Theories, Instantons.

Contents

1. Introduction

Recently Alday Gaotto and Tachikawa [1] have found a very remarkable relation between generalized partition functions [2, 3, 4, 5] in certain classes of $\mathcal{N}=2$ conformal $SU(2)$ quiver gauge theories [6] (see [7] for instanton counting in quiver theories) and correlation functions of 2d Liouville theory on Riemann surfaces. In this paper I will consider only the "holomorphic" version of AGT conjecture concerning the relation between instanton part of the partition function in gauge theory and the conformal block in 2d CFT. More precisely I'll concentrate on $SU(2)$ gauge theories with four fundamental hypermultiplets and the theory with an adjoint hypermultiplet. This choice is of particular interest since they are related to such fundamental objects of 2d CFT as 4-point and torus 1-point conformal blocks. I will adopt in this paper a slightly generalized version of the original AGT conjecture and will not assume a specific relation between Nekrasov's deformation parameters ϵ_1 and ϵ_2, a point emphasized also in recent works [8, 9]. Another deviation from the initial AGT will be discussion of non-conformal gauge theories (a possibility also discussed in very recent papers [10, 9] from a different perspective).

In all further discussions the recursion relation for the CFT 4-point conformal block discovered by Alexei Zamolodchikov [11] a quarter of century ago will play the central role. Unfortunately this brilliant work is not widely known even by the specialists.

The section 2. is a breaf introduction to the Zamolodchikov's recursion relation.

The section 3. is devoted to the description of the instanton part of the generalized partition function in $\mathcal{N} = 2$ SYM theories introdced by Nekrasov [2]. Representation of the Nekrasov partition function as a sum over (multiple) Young tableau in a way suitable for practical higher order instanton calculations is based on the character formula describing decomposition of the tangent space of the moduli space of instantons under the combined (global gauge plus space-time rotations) torus action around fixed points [3]. The Nekrasov partition function for the cases with different types of extra hypermultiplets can be read of from the character formula incorporating specific factors which depend on the representations of the hypermultiplets [2, 4, 5]. In this section for further reference the cases of fundamental or adjoint hypermultiplets are presented in some details.

In section 4. the Zamolodcikov recursion relation for 4-point conformal block is translated into the relation for the partition function with four fundamentals. As a particular application the exact large vev asymptotic of the partition function is derived and the leading term is checked against the Seiberg-Witten curve [12] analysis. Considering the large mass limit when one or more fundamentals decouple the analogous relations for less number of hypermultiplets are derived. The recursion relation for the case without extra matter is especially simple and may serve as a convenient starting point for investigation of the analytical properties of the prepotential in presence of nonzero gravitational background.

In section 5. a similar recursion relation is proposed for the case of adjoint hypermultiplet. The conjectured relation has been checked against explicit instanton calculations up to order 5. Again the AGT conjecture leads to analogous (previously unknown in CFT) relation for the torus 1-point conformal block. Comparison of the results of this sections with those of previous one leads to a surprising conclusion: the torus 1-point block is closely related to specific sphere 4-point block or alternatively the generalized $\mathcal{N} = 2$ SYM partition function with adjoint hypermultiplet is related to the partition function with four hypermultiplets.

Finally the appendix A. briefly describes how to calculate the torus one point conformal block from CFT first principles.

2. Zamolodchikov's q-recursion relation for CFT conformal block

Though there is no closed analytic expression for the general 4-point conformal block,

Al. Zamolodchikov has found an extremely convenient recursion (Russian doll type) relation, which allows to calculate the conformal block up to the desired order in x-expansion. Below I give a brief description of Zamolodchikov's recursion relation closely following to [13]. It is convenient to represent the generic 4-point conformal block $\mathcal{F}$ as [11]

$$\begin{aligned}\mathcal{F}(\Delta_i, \Delta, x) = (16q)^{-\alpha^2} x^{Q^2/4-\Delta_1-\Delta_2}(1-x)^{Q^2/4-\Delta_1-\Delta_3} \\ \times\theta_3(q)^{3Q^2-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)} H(\mu_i, \Delta, q)\end{aligned} \tag{2.1}$$

where

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.2}$$

Here Δ_i, $i = 1, 2, 3, 4$ - the dimensions of the external (primary) fields (placed at the points x, 0, 1 and ∞ respectively) and Δ - the internal dimension are parametrized by

$$\Delta_i = \frac{Q^2}{4} - \lambda_i^2 \,, \qquad \Delta = \frac{Q^2}{4} - \alpha^2 \,, \tag{2.3}$$

where Q is related to the central charge of the Virasoro algebra through

$$c = 1 - 6Q^2 \tag{2.4}$$

For further purposes I parametrise the background charge Q via

$$Q = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}}. \tag{2.5}$$

and introduce the parameters μ_i (later to be related with the masses of the anti-fundamental hyper-multiplets under the AGT conjecture) as linear combinations of λ_i:

$$\begin{aligned}\mu_1 = \lambda_1 + \lambda_2 + \frac{Q}{2}\,,\ \mu_2 = \lambda_1 - \lambda_2 + \frac{Q}{2}\,, \\ \mu_3 = \lambda_3 + \lambda_4 + \frac{Q}{2}\,,\ \mu_4 = \lambda_3 - \lambda_4 + \frac{Q}{2}\,.\end{aligned} \tag{2.6}$$

Comparing with the standard $Q = b + 1/b$ we see that $b = \sqrt{\epsilon_1/\epsilon_2}$.[1] The parameter $q = e^{i\pi\tau}$ is related to the coordinate x:

$$\tau = i\frac{K(1-x)}{K(x)}, \tag{2.7}$$

[1] In fact upon simple rescaling of the masses and vev's of a $\mathcal{N} = 2$ conformal SYM theory by the factor $1/\sqrt{\epsilon_1\epsilon_2}$, all expressions become homogeneous in $\epsilon_{1,2}$. Thus there is no need to follow [1] and restrict oneself to the case $\epsilon_1\epsilon_2 = 1$.

where

$$K(x) = \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}} \; . \tag{2.8}$$

Conversely

$$x = \frac{\theta_2^4(q)}{\theta_3^4(q)}, \tag{2.9}$$

where

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}. \tag{2.10}$$

Here are the first few terms of the small x expansion of q:

$$16q = x + \frac{x^2}{2} + \frac{21x^3}{64} + \frac{31x^4}{128} + \mathcal{O}(x^5) \tag{2.11}$$

The asymptotic behaviour of the conformal block at large internal dimension $\Delta \to \infty$ has been established in [11] which in terms of the function H is very simple:

$$H = 1 + \mathcal{O}(\Delta) \tag{2.12}$$

Now we are ready to state Al.Zamolodchikov's q-recursion relation:

$$H(\mu_i, \Delta, q) = 1 + \sum_{m,n=1}^{\infty} \frac{q^{mn} R_{m,n}^{(4)}}{\Delta - \Delta_{m,n}} H(\mu_i, \Delta_{m,n} + mn, q), \tag{2.13}$$

where the poles are located at

$$\Delta_{m,n} = \frac{Q^2}{4} - \lambda_{m,n}^2 \tag{2.14}$$

with

$$\lambda_{m,n} = \frac{m\epsilon_1 + n\epsilon_2}{2\sqrt{\epsilon_1\epsilon_2}} \tag{2.15}$$

i.e. exactly at the degenerated internal dimensions. Finally

$$R_{m,n}^{(f)} = \frac{2\prod_{r,s}\prod_{i=1}^{f}(\mu_i - \frac{Q}{2} - \lambda_{r,s})}{\prod'_{k,l}\lambda_{k,l}}, \tag{2.16}$$

where the products are over the pairs (r, s) and (k, l) within the range

$$\begin{aligned} r &= -m+1, -m+3, \ldots, m-1 \\ s &= -n+1, -n+3, \ldots, n-1 \\ k &= -m+1, -m+2, \ldots, m-1, m \\ l &= -n+1, -n+2, \ldots, n-1, n \end{aligned}$$

while prime over the product in the denominator indicates that the pairs (m,n) and $(0,0)$ should be suppressed.

Note that the function H in order to satisfy (2.13) should be totally symmetric w.r.t. permutations of μ_i's. Less obvious is the symmetry with respect to reflections $\mu_i \to Q - \mu_i$ accompanied with $q \to -q$ which is a consequence of $\lambda_{-r,-s} = -\lambda_{r,s}$. There is no doubt that these remarkably reach symmetries of the 4-point block and their consequences are worth to be explored in greater details.

In order to give some flavor how the recursion relation (2.13) works in practice I give the result of iteration up to the order q^2

$$H = 1 + \frac{R_{1,1}^{(f)} q}{\Delta - \Delta_{1,1}} + \left(\frac{(R_{1,1}^{(f)})^2}{\Delta - \Delta_{1,1}} + \frac{R_{1,2}^{(f)}}{\Delta - \Delta_{1,2}} + \frac{R_{2,1}^{(f)}}{\Delta - \Delta_{2,1}} \right) q^2 + \mathcal{O}(q^3) \quad (2.17)$$

Another elementary but useful observation is that at the order q^l one encounters poles at $\Delta = \Delta_{n,m}$ with $nm < l$.

3. Generalized partition function of $\mathcal{N} = 2$ SYM with fundamental or adjoint hyper-multiplets

In the seminal paper [2] Nekrasov has proposed to generalize the Seiberg-Witten prepotential including into the game besides unbroken gauge transformation also the space time rotations which allowed to localize instanton contributions around finite number of fixed points. The general problem of computing the contribution of a given fixed point has found its final solution in [3]. When the gauge group is $U(N)$ the fixed points are in one to one correspondence with the arrays of Young tableau $\vec{Y} = (Y_1, ..., Y_N)$ with total number of boxes $|\vec{Y}|$ being equal to the instanton charge k. The (holomorphic) tangent space of the moduli space of instantons decomposes into sum of (complex) one dimensional irreducible representations of the Cartan subgroup of $U(N) \times O(4)$ [3]

$$\chi = \sum_{\alpha,\beta=1}^{N} e_\beta e_\alpha^{-1} \left\{ \sum_{s \in Y_\alpha} \left(T_1^{-l_{Y_\beta}(s)} T_2^{a_{Y_\alpha}(s)+1} \right) + \sum_{s \in Y_\beta} \left(T_1^{l_{Y_\alpha}(s)+1} T_2^{-a_{Y_\beta}(s)} \right) \right\}, \quad (3.1)$$

where $(e_1, ..., e_N) = (e^{ia_1}, \ldots, e^{ia_N}) \in U(1)^N \subset U(N)$ and $(T_1, T_2) = (e^{i\epsilon_1}, e^{i\epsilon_2}) \in U(1)^2 \subset O(4)$, $a_Y(s)$ $(l_{Y_\alpha}(s))$ is the distance of the right edge of the box s from the limiting polygonal curve of the Young tableaux Y in horizontal (vertical) direction taken with the sign plus if the box $s \in Y_\alpha$ and with the sign minus otherwise.

One-dimensional subgroups of the above mentioned $N+2$ dimensional torus are generated by the vector fields parametrized by $a_1, \ldots, a_N$ and ϵ_1, ϵ_2. From the physical point of view a_α are the vacuum expectation values of the complex scalar of the $\mathcal{N} = 2$ gauge multiplet and ϵ_1, ϵ_2 specify a particular gravitational background now

commonly called Ω-background. The contribution of a fixed point to the Nekrasov partition function in the basic $\mathcal{N}=2$ case without extra hypermultiplets is simply the inverse determinant of the above mentioned vector field action on the tangent space at given fixed point. All the eigenvalues of this vector field can be directly read off from the character formula (3.1) . The result is [3]

$$P_{gauge}(\vec{Y}) = \prod_{\alpha,\beta=1}^{N} \prod_{s\in Y_\alpha} \frac{1}{E_{\alpha,\beta}(s)(\epsilon - E_{\alpha,\beta}(s))}, \tag{3.2}$$

where

$$E_{\alpha,\beta} = a_\beta - a_\alpha - \epsilon_1 l_{Y_\beta}(s) + \epsilon_2(a_{Y_\alpha}(s)+1) \tag{3.3}$$

In general the theory may include "matter" hypermultiplets in various representations of the gauge group. In that case one should multiply the gauge multiplet contribution (3.2) by another factor P_{matter}. In this paper we will consider the case of several (up to four) hypermultiplets in anti-fundamental representation and also the theory with an adjoint hypermultiplet (so called $\mathcal{N}=2^*$). The respective matter factors read [4]

$$P_{antifund}(\vec{Y}) = \prod_{l=1}^{f}\prod_{\alpha=1}^{N}\prod_{s_\alpha\in Y_\alpha}(\chi_{\alpha,s_\alpha}+m_l) \tag{3.4}$$

$$P_{adj}(\vec{Y}) = \prod_{\alpha,\beta=1}^{N}\prod_{s\in Y_\alpha}(E_{\alpha,\beta}(s)-M)(\epsilon - E_{\alpha,\beta}(s)-M), \tag{3.5}$$

where m_l, M are the masses of the hypermultiplets, $\epsilon = \epsilon_1+\epsilon_2$,

$$\chi_{\alpha,s_\alpha} = a_\alpha + (i_{s_\alpha}-1)\epsilon_1 + (j_{s_\alpha}-1)\epsilon_2 \tag{3.6}$$

and i_{s_α}, j_{s_α} are the numbers of the column and the row of the tableaux Y_α where the box s_α is located. Note also that in order to get a fundamental hypermultiplet instead of an antifundamental, one should simply replace the respective mass m_l by $\epsilon - m_l$ in (3.4). In terms of above defined quantities the instanton part of Nekrasov partition function reads[2]

$$Z_{inst} = \sum_{\vec{Y}} x^{|\vec{Y}|} P_{gauge}(\vec{Y}) P_{matter}(\vec{Y}) \tag{3.7}$$

4. AGT conjecture for the four-point conformal block and recursion relations for $\mathcal{N}=2$ SYM with extra fundamentals

From now on we will consider only the gauge group $SU(2)$ and will set the vacuum expectation values $a_1 = -a_2 = a$.

[2]I use notation $x = e^{2\pi i \tau_g}$ with τ_g the usual gauge theory coupling to avoid confusion with the already introduced in chapter 2 parameter q and to make comparison with 2d CFT block transparent.

4.1 f=4 antifundamentals

According to the AGT conjecture for the case of $f = 4$ extra antifundamental hypermultiplets one has [1]

$$Z_{inst}^{(4)}(a, m_i, x) = x^{\Delta_1+\Delta_2-\Delta}(1-x)^{2(\lambda_1+\frac{Q}{2})(\lambda_3+\frac{Q}{2})}\mathcal{F}(\Delta, \Delta_i, x), \tag{4.1}$$

where $Z_{inst}^{(4)}(a, m_i, x)$ is given by (3.4), (3.7) specialized to the case of the gauge group $SU(2)$ and $f = 4$ flavours. The vev $a = \alpha\sqrt{\epsilon_1\epsilon_2}$ and masses $m_i = \mu_i\sqrt{\epsilon_1\epsilon_2}$ are related to the conformal dimensions Δ, Δ_i through (2.3), (2.6). Equivalently, taking into account (2.1):

$$\begin{aligned} Z_{inst}^{(4)}(a, m_i, x) = &\left(\frac{x}{16q}\right)^{\alpha^2} (1-x)^{\frac{1}{4}(Q-\sum_{i=1}^4 \mu_i)^2} \\ &\times [\theta_3(q)]^{2\sum_{i=1}^4(\mu_i^2-Q\mu_i)+Q^2}\, H(\mu_i, \Delta, q). \end{aligned} \tag{4.2}$$

Thus Zamolodchikov's recursion relation (2.13) for four point conformal block automatically provides a very efficient tool also for calculating the partition function of the $SU(2)$ SYM theory with extra four (anti) fundamental hypermultiplets. One obvious advantage of the recursion relation compared to the explicit formula (3.7) is that at each order of the "renormalized" (through the relation (2.9)) instanton parameter q the former immediately determines the pole structure in variable $\Delta = Q^2/4 - \alpha^2$ (see remark after Eq. (2.17)). Alternatively the formula (3.7) is a sum over all (rapidly growing number of) couples of Young tableau each term being a simple factorized rational expression. Unfortunately the poles of the individual terms are extremely redundant: most of the poles after summation of all terms of given instanton order disappear. In this sense the Zamolodchikov recursion relation and the explicit formula (3.7) play complementary roles: the recursion relation provides a powerful tool for investigation of the analytical properties of the Nekrasov partition function while the Eq. (3.7) together with Eq. (4.2) provide a closed expression for the four point conformal block. Needless to say both tasks are of considerable importance and were waiting long time to find a solution.

As a most immediate consequence of Eq. (4.2) one learns the asymptotic behaviour of the partition function at large vev's

$$\begin{aligned} Z_{inst}^{(4)}(a, m_i, x) \sim &\left(\frac{x}{16q}\right)^{\frac{a^2}{\epsilon_1\epsilon_2}} (1-x)^{\frac{1}{4\epsilon_1\epsilon_2}\left(\epsilon-\sum_{i=1}^4 m_i\right)^2} \\ &\times [\theta_3(q)]^{\frac{1}{\epsilon_1\epsilon_2}\sum_{i=1}^4\left(m_i^2+(\epsilon-m_i)^2-3\epsilon^2/4\right)}. \end{aligned} \tag{4.3}$$

The instanton part of the Seiberg-Witten prepotential is

$$F_{inst}^{SW} = -\lim_{\epsilon_{1,2}\to 0}\epsilon_1\epsilon_2 Z_{inst} \tag{4.4}$$

hence

$$F_{inst}^{SW} \sim a^2 \log \frac{16q}{x} - \frac{1}{4}\left(\sum_{i=1}^{4} m_i\right)^2 \log(1-x) - 2\sum_{i=1}^{4} m_i^2 \log\theta_3(q) \tag{4.5}$$

Taking into account Eq. (2.9), (2.11) for the first leading in a^2 term one gets

$$F_{inst}^{SW} \sim a^2 \log \frac{16q}{x} = a^2\left(\frac{x}{2} + \frac{13x^2}{64} + \frac{23x^3}{192} + \frac{2701x^4}{32768} + \frac{5057x^5}{81920} + \cdots\right) \tag{4.6}$$

which coincides with the expression given in part B3 of [1]. It would be interesting to extract the subleading m corrections directly from the Seiberg-Witten curve too.

4.2 $f = 3$ antifundamentals

We have already seen how fruitful is the incorporation of AGT conjecture with Zamolodchikov's recursion relation. In order to get recursion relations also for less number (or even without) hypermultiplets one can decouple the extra hypermultiplets one after another by sending the masses to infinity. It is obvious from the Eq. (3.4), (3.7) that to decouple one of the hypermultiplets (say the one with mass m_4) one should renormalise the instanton parameter $x \to x/m_4$ and go to the limit $m_4 \to \infty$. Similarly examining Zamolodchikov's recursion relation (2.13) we see that there exists a smooth limit for the function H at large μ_4, provided one simultaneously redefines the parameter $q \to q/m_4 = q/(\mu_4\sqrt{\epsilon_1\epsilon_2})$. Indeed for large μ_4 limit $R_{m,n}^{(4)} \sim \mu_4^{mn} R_{m,n}^{(3)}$. It remains to investigate the behaviour of the prefactors of the function H in (4.2). The analysis is elementary and boils down to expanding $\theta_3(q)$ up to first order: $\theta_3 = 1 + 2q + \mathcal{O}(q^2)$ and taking into account the relation (2.11) between q and x (keeping first two terms is enough). Here is the result

$$Z_{inst}^{(3)}(a, m_i, x) = e^{-\frac{x}{4\epsilon_1\epsilon_2}\left(\frac{x}{16} - \epsilon + 2m_1 + 2m_2 + 2m_3\right)} H^{(3)}(\mu_i, \Delta, q), \tag{4.7}$$

where $q = \frac{x}{16\sqrt{\epsilon_1\epsilon_2}}$. Again at large Δ the function $H^{(3)}(\mu_i, \Delta, q) \sim 1$ and satisfies the recursion relation (2.13) with $R_{m,n}^{(4)}$ replaced by $R_{m,n}^{(3)}$ (see (2.16)).

4.3 Theories with $f = 2, 1$ or 0

It is straightforward to repeat the procedure of previous subsection and decouple more (or even all) hypermultiplets.

- $f = 2$

$$Z_{inst}^{(2)}(a, m_1, m_2, x) = e^{-\frac{x}{2\epsilon_1\epsilon_2}} H^{(2)}(\mu_1, \mu_2, \Delta, q), \tag{4.8}$$

with $q = \frac{x}{16\epsilon_1\epsilon_2}$.

- $f = 1$

$$Z^{(1)}_{inst}(a, m_1, x) = H^{(1)}(\mu_1, \Delta, q), \tag{4.9}$$

now with $q = \frac{x}{16(\epsilon_1\epsilon_2)^{3/2}}$.

- $f = 0$

$$Z_{inst}(a, x) = H^{(0)}(\Delta, q), \tag{4.10}$$

and $q = \frac{x}{16(\epsilon_1\epsilon_2)^2}$.

The recursion relation for the pure $\mathcal{N} = 2$ theory is especially simple and it is worth to rewrite it here in intrinsic terms:

$$Z_{inst}(a, x) = 1 - \sum_{m,n=1}^{\infty} \frac{x^{mn}\mathcal{R}_{m,n}}{4a^2 - (m\epsilon_1 + n\epsilon_2)^2} \, Z_{inst}((m\epsilon_1 - n\epsilon_2)/2, x), \tag{4.11}$$

where

$$\mathcal{R}_{m,n} = 2 \prod_{k,l}{}' (k\epsilon_1 + l\epsilon_2)^{-1}, \tag{4.12}$$

and the range of the product over k, l is the same as in Eq. (2.16).

It has been shown in recent papers [10, 9] that when the the number of fundamental hypermultiplets $f < 4$ on CFT side one has irregular conformal blocks.

5. Recursion relation for the case with adjoint hypermultiplet

Encouraged with the success in the cases with fundamental hypermultiplets, it is natural to expect that a recursion relation of the same kind should exist also for the case of adjoint hypermultiplet or due to AGT conjecture for the torus 1-point conformal block. In fact, explicit computation in first few orders of instanon expansion of the generalized partition function with adjoint and investigation of their large a^2 behaviour together with some intuition gained from the previous examples leads to the desired result. Define the function H_{tor} by

$$Z^{(adj)}_{inst}(a, M, q) = [\hat{\eta}(q)]^{\frac{-2(M-\epsilon_1)(M-\epsilon_2)}{\epsilon_1\epsilon_2}} H_{tor}(\mu, \Delta, q) \tag{5.1}$$

where

$$\hat{\eta}(q) = \prod_{n=1}^{\infty}(1 - q^n) \tag{5.2}$$

(we set $\mu = \frac{2M-\epsilon}{2\sqrt{\epsilon_1\epsilon_2}}$, $\Delta = \frac{Q^2}{4} - \alpha^2$, $\alpha = \frac{a}{\sqrt{\epsilon_1\epsilon_2}}$ and instead of x restore the conventional notation q for instanton parameter). Then $H_{tor} = 1 + \mathcal{O}(\Delta)$ satisfies the relation

$$H_{tor}(\mu, \Delta, q) = 1 + \sum_{m,n=1}^{\infty} \frac{q^{mn} R_{m,n}^{(tor)}}{\Delta - \Delta_{m,n}} H_{tor}(\mu, \Delta_{m,n} + mn, q), \tag{5.3}$$

where

$$R_{m,n}^{(tor)} = R_{m,n}^{(4)} \tag{5.4}$$

As earlier $R_{m,n}^{(4)}$ is given by Eq. (2.16) but the four parameters μ_i are specified as

$$\mu_1 = \frac{M}{2\sqrt{\epsilon_1\epsilon_2}}; \quad \mu_2 = \frac{M+\epsilon_1}{2\sqrt{\epsilon_1\epsilon_2}}; \quad \mu_3 = \frac{M+\epsilon_2}{2\sqrt{\epsilon_1\epsilon_2}}; \quad \mu_4 = \frac{M+\epsilon_1+\epsilon_2}{2\sqrt{\epsilon_1\epsilon_2}} \tag{5.5}$$

I have checked the conjecture (5.1), (5.3) up to 5 instantons. The prefactor of H_{tor} in (5.1) defines the large a^2 behaviour of Z_{adj} and hence that of the prepotential

$$F_{inst,adj}^{SW} \sim M^2 \log \hat{\eta}(q) \tag{5.6}$$

in agreement with the result derived from Seiberg-Witten curve (see e.g. [14]).

Incorporating above results with AGT conjecture for adjoint hypermultiplet we find the equivalent recursion relation for torus 1-point conformal block

$$\mathcal{F}_\alpha^\mu(q) = [\hat{\eta}(q)]^{-1} H_{tor}(\mu, \Delta, q) \tag{5.7}$$

Observe that the recursion relation (2.13) together with asymptotic condition (2.12) uniquely determines H in terms of $R_{m,n}$. Thus the Eq's. (5.3), (5.4) and (5.5) lead to conclusion that

$$H_{tor}(\mu, \Delta, q) = H(\mu_1, \mu_2, \mu_3, \mu_4, \Delta, q) \tag{5.8}$$

provided the relations (5.5) are hold. This is a very exciting result: the torus 1-point block is closely related to specific sphere 4-point block on sphere or alternatively the generalized $\mathcal{N} = 2$ SYM partition function with four hypermultiplets is related to the partition function with adjoint hypermultiplet. On CFT side e.g. one gets (in the reminder of this section the relation (2.9) between parameters q and x is always assumed)

$$\begin{aligned}\mathcal{F}_\alpha^\mu(q) = [\hat{\eta}(q)]^{-1} (16q)^{\alpha^2} x^{-Q^2/4+\Delta_1+\Delta_2} \\ \times (1-x)^{-Q^2/4+\Delta_1+\Delta_3} \theta_3(q)^{-3Q^2+4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)} \mathcal{F}(\Delta_i, \Delta, x)\end{aligned} \tag{5.9}$$

Owing to already mentioned reach symmetry of the function $\mathcal{F}$ for given dimensions $\Delta = Q^2/4 - \alpha^2$ and $\Delta_\mu = Q^2/4 - \mu^2$ the choice of $\Delta_i = Q^2/4 - \lambda_i^2$ is not unique. The essentially different choices are

$$\lambda_1 = \frac{\mu}{2}; \quad \lambda_2 = \frac{Q}{4}; \quad \lambda_3 = \frac{\mu}{2}; \quad \lambda_4 = \frac{1}{4}\sqrt{Q^2 - 4} \tag{5.10}$$

and somewhat more sophisticated

$$\lambda_1 = \frac{\mu}{2} - \frac{1}{8}\left(Q + \sqrt{Q^2-4}\right) ; \quad \lambda_2 = \frac{1}{8}\left(Q - \sqrt{Q^2-4}\right) ;$$
$$\lambda_3 = \frac{\mu}{2} + \frac{1}{8}\left(Q + \sqrt{Q^2-4}\right) ; \quad \lambda_4 = \frac{1}{8}\left(Q - \sqrt{Q^2-4}\right) . \tag{5.11}$$

Since the calculation of the torus 1-point function $\mathcal{F}_\alpha^\mu(q)$ from first principles of CFT is less familiar for reader's convenience I sketch the procedure in the appendix.

The Eq.s (4.1), (5.1) and (5.7) straightforwardly lead to analogues (and by no means less surprising) relation among SYM partition function with four hypermultiplets and the partition function with adjoint hypermultiplet

$$Z_{inst}^{(adj)}(a,M,q) = [\hat{\eta}(q)]^{\frac{-2(M-\epsilon_1)(M-\epsilon_2)}{\epsilon_1\epsilon_2}} \left(\frac{x}{16q}\right)^{-\frac{a^2}{\epsilon_1\epsilon_2}} (1-x)^{-\frac{1}{4\epsilon_1\epsilon_2}(\epsilon-\sum_{i=1}^4 m_i)^2}$$
$$\times\ [\theta_3(q)]^{-\frac{1}{\epsilon_1\epsilon_2}\sum_{i=1}^4\left(m_i^2+(\epsilon-m_i)^2-3\epsilon^2/4\right)} Z_{inst}^{(4)}(a,m_i,x) \tag{5.12}$$

Here the masses $m_i = \sqrt{\epsilon_1\epsilon_2}\,\mu_i$ (up to permutations) are given by Eq. (5.5).

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Note added

After the first version of this paper appeared in arXiv Vl. Fateev, A. Litvinov and S. Ribault kindly informed me about the work [16] where a relation between the 1-point correlation function on the torus and a specific 4-point correlation function on sphere is established. Contrary to the one, suggested in this article, that relation holds for different (though simply related) values of the central charge on the sphere and torus. To avoid confusion let me note also that in present paper the modular parameter of the torus is related to the parameter q introduced in section 2. via $q = \exp(2\pi i \tau_{tor})$, *i.e.* $\tau_{tor} = \tau/2$ with τ given by Eq. (2.8) while the modular parameter of the Ref. [16] is equal to τ. Naturally the relation of Ref. [16] between correlation functions boils down to a certain relation among conformal blocks. The

condition that this relation and the one conjectured in present paper are compatible is equivalent to the following non-trivial identity (see Eq. (2.1) for definition of the function H and the Eq.'s (2.6), (2.3) which relate its arguments μ_i to the dimensions of external fields):

$$H_{\tilde{b}}(\tilde{\mu}_i, \tilde{\Delta}, q^2) = H_b(\mu_i, \Delta, q)$$

where the dependence on the parameter b specifying the central charge is indicated explicitly and the remaining parameters are specified as (the parameters established in present paper are on the first line while those on second line are found in [16]):

$$\tilde{\mu}_1 = \frac{\tilde{\mu}}{2} + \frac{\tilde{b}}{4} + \frac{1}{4\tilde{b}}; \quad \tilde{\mu}_2 = \frac{\tilde{\mu}}{2} + \frac{3\tilde{b}}{4} + \frac{1}{4\tilde{b}}; \quad \tilde{\mu}_3 = \frac{\tilde{\mu}}{2} + \frac{\tilde{b}}{4} + \frac{3}{4\tilde{b}}; \quad \tilde{\mu}_4 = \frac{\tilde{\mu}}{2} + \frac{3\tilde{b}}{4} + \frac{3}{4\tilde{b}}$$
$$\mu_1 = \mu + \frac{b}{4} + \frac{1}{2b}; \quad \mu_2 = \mu + \frac{3b}{4} + \frac{1}{2b}; \quad \mu_3 = b + \frac{1}{2b}; \quad \mu_4 = \frac{b}{2} + \frac{1}{2b}$$

and

$$\tilde{b} = \frac{b}{\sqrt{2}}; \quad \tilde{\mu} = \sqrt{2}\mu; \quad \tilde{\Delta} = \frac{(\tilde{b} + \frac{1}{\tilde{b}})^2}{4} - \tilde{\alpha}^2; \quad \Delta = \frac{(b + \frac{1}{b})^2}{4} - \alpha^2; \quad \tilde{\alpha} = \frac{\alpha}{\sqrt{2}}$$

It should be possible to find a mathematical proof for this identity based on Zamolodchikov recursion relation (2.13). Direct calculation using the recursion relation (2.13) confirms that the identity indeed holds up to high orders in q.

A. Torus 1-point block

Below is presented the calculation of the one-point conformal block on torus $tr_\alpha\left(q^{L_0 - c/12}\phi_\mu\right)$ up to level 2 (in principle the computation can be carried out up to arbitrary level). Start with (chiral) OPE

$$\begin{aligned}\phi_\mu(x)\phi_\alpha(0) = \sum_Y x^{-\Delta_\mu + |Y|}\beta_{\mu\alpha}^{\alpha Y} L_{-Y}\phi_\alpha(0) = \\ x^{-\Delta_\mu}(1 + x\beta^1 L_{-1} + x^2(\beta^{11}L_{-1}^2 + \beta^2 L_{-2}) + \cdots)\phi_\alpha(0)\end{aligned} \tag{A.1}$$

where for the partition $Y = \{k_1 \geq k_2 \geq \cdots \geq 0\}$, $|Y| = k_1 + k_2 + \cdots$,

$$L_{-Y} \equiv L_{-k_1}L_{-k_2}\cdots \tag{A.2}$$

and L_k are the standard Virasoro generators. It is well known that the coefficients β in principle could be calculated level by level using conformal symmetry [15]. In particular the first few coefficients β explicitly presented in (A.1) are

$$\begin{aligned}\beta^{\{1\}} &= \frac{\Delta_\mu}{2\Delta} \\ \beta^{\{11\}} &= \frac{\Delta_\mu\left(c - 16\Delta + (c + 8\Delta)\Delta_\mu\right)}{4\Delta(c + 2c\Delta + 2\Delta(-5 + 8\Delta))}, \\ \beta_2^{\{2\}} &= \frac{(1 + 8\Delta - 3\Delta_\mu)\,\Delta_\mu}{c + 2c\Delta + 2\Delta(-5 + 8\Delta)}\end{aligned} \tag{A.3}$$

To calculate the trace the diagonal matrix elements of the OPE of the primary field ϕ_M with the descendants of the field ϕ_α are needed. Using the commutation relation

$$[L_n, \phi_\mu(x)] = x^n(x\partial + (1+n)\Delta_\mu)\phi_\mu(x) \tag{A.4}$$

one easily finds

$$\begin{aligned}
\phi_\mu(x)L_{-1}\phi_\alpha(0) &= x^{-\Delta_\mu}(x^{-1}\Delta_\mu + (1+\beta^1(\Delta_\mu - 1))L_{-1} + \cdots)\phi_\alpha(0) \\
\phi_\mu(x)L_{-2}\phi_\alpha(0) &= x^{-\Delta_\mu}(x^{-1}\Delta_\mu + (1+\beta^1(\Delta_\mu - 1))L_{-1} + \cdots)\phi_\alpha(0) \\
\phi_\mu(x)L_{-1}^2\phi_\alpha(0) &= x^{-\Delta_\mu}(x^{-2}(\Delta_\mu - 1)(\Delta_\mu + 1)(\beta^{11}L_{-1}^2 + \beta^2 L_{-2}) \\
&\quad + x^{-1}2(\Delta_\mu - 1)L_{-1} + L_{-1}^2 + \cdots)\phi_\alpha(0)
\end{aligned} \tag{A.5}$$

It remains to read off the diagonal matrix elements from (A.5) to get the torus one point block

$$\begin{aligned}
\mathcal{F}_\alpha^\mu(q) &\equiv q^{-\Delta_\alpha + \frac{c}{12}} tr_\alpha \left(q^{L_0 - c/12}\phi_\mu(1)\right) = 1 + \left(1 + (\Delta_\mu - 1)\,\beta^{\{1\}}\right) q \\
&+ \left((\Delta_\mu - 1)\,(\Delta_\mu - 2)\,\beta^{\{11\}} 2\,(\Delta_\mu - 1)\,\beta^{\{1\}} + 2\,(\Delta_\mu - 1)\,\beta^{\{2\}} + 2\right) q^2 + \cdots
\end{aligned} \tag{A.6}$$

It is easy to check that this result is consistent with recursion relation (5.3), (5.7).

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